

Generalized line graphs: Cartesian products and complexity of recognition *

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Abstract

Putting the concept of line graph in a more general setting, for a positive integer k the k -line graph $L_k(G)$ of a graph G has the K_k -subgraphs of G as its vertices, and two vertices of $L_k(G)$ are adjacent if the corresponding copies of K_k in G share $k - 1$ vertices. Then, 2-line graph is just the line graph in usual sense, whilst 3-line graph is also known as triangle graph. The k -anti-Gallai graph $\triangle_k(G)$ of G is a specified subgraph of $L_k(G)$ in which two vertices are adjacent if the corresponding two K_k -subgraphs are contained in a common K_{k+1} -subgraph in G .

We give a unified characterization for nontrivial connected graphs G and F such that the Cartesian product $G \square F$ is a k -line graph. In particular for $k = 3$, this answers the question of Bagga (2004), yielding the necessary and sufficient condition that G is the line graph of a triangle-free graph and F is a complete graph (or vice versa). We show that for any $k \geq 3$, the k -line graph of a connected graph G is isomorphic to the line graph of G if and only if $G = K_{k+2}$. Furthermore, we prove that the recognition problem of k -line graphs and that of k -anti-Gallai graphs are NP-complete for each $k \geq 3$.

Keywords: Triangle graph, k -line graph, anti-Gallai graph, Cartesian product graph.

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1 Introduction

The line graph $L(G)$ of a graph G has vertices representing the edges (K_2 -subgraphs) of G and two vertices in the line graph are adjacent if and only if the corresponding edges share a vertex (a K_1 subgraph) in G . The analogous notion in the dimension higher by one is the triangle graph $\mathcal{T}(G)$ of G whose vertices correspond to the triangles (K_3 -subgraphs) of G and the vertices representing triangles having a common edge (K_2 -subgraph) are adjacent. The natural generalization gives the notion of k -line graph, which together with its specified subgraph, the so-called k -anti-Gallai graph is the main subject of this paper.

1.1 Terminology

All graphs considered here are simple and undirected. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Throughout this paper, a k -clique of G will be meant as a complete $K_k \subseteq G$ subgraph. That is, *inclusion-wise maximality is not required*. For the sake of simplicity, if the meaning is clear from the context, we do not distinguish between a clique and its vertex set in notation (e.g., the vertex set of a clique C will also be denoted by C instead of $V(C)$). The *clique number* $\omega(G)$ is the maximum order of a clique contained in G . The *Cartesian product* of two graphs G and F , denoted by $G \square F$, has the ordered pairs (u, v) as its vertices where $u \in V(G)$ and $v \in V(F)$, and two vertices (u, v) and (u', v') are adjacent if $u = u'$ and v is adjacent to v' or $v = v'$ and u is adjacent to u' . If $v_i \in V(F)$, the *copy* G_i is the subgraph of $G \square F$ induced by the vertex set $V(G_i) = \{(u_j, v_i) : u_j \in V(G)\}$. The copy F_j for $u_j \in V(G)$ is meant similarly. The *join* $G \vee F$ of two vertex-disjoint graphs is the graph whose vertex set is $V(G) \cup V(F)$ and two vertices u and v of $G \vee F$ are adjacent if and only if either $uv \in E(G) \cup E(F)$, or $u \in V(G)$ and $v \in V(F)$. The *diamond* is a 4-cycle with exactly one chord (or equivalently, the graph $K_4 - e$ obtained from the complete graph K_4 by deleting exactly one edge). Given a graph F , a graph G is said to be F -free if it contains no *induced* subgraph isomorphic to F .

Next, we define the two main concepts studied in this paper. For illustration, see Figure 1.

Definition 1. For an integer $k \geq 1$, the k -line graph $L_k(G)$ of a graph G has vertices representing the k -cliques of G , and two vertices in $L_k(G)$ are adjacent if and only if the represented k -cliques of G intersect in a $(k - 1)$ -clique.

For $k = 1$ the definition yields $L_1(G) = K_n$ for every graph G of order n . Note that even the K_2 -free (edgeless) graph with n vertices has the complete graph K_n as its 1-line graph. The 2-line graph $L_2(G)$ is the line graph of G in the usual sense. The 3-line graph is the triangle graph $\mathcal{T}(G)$.

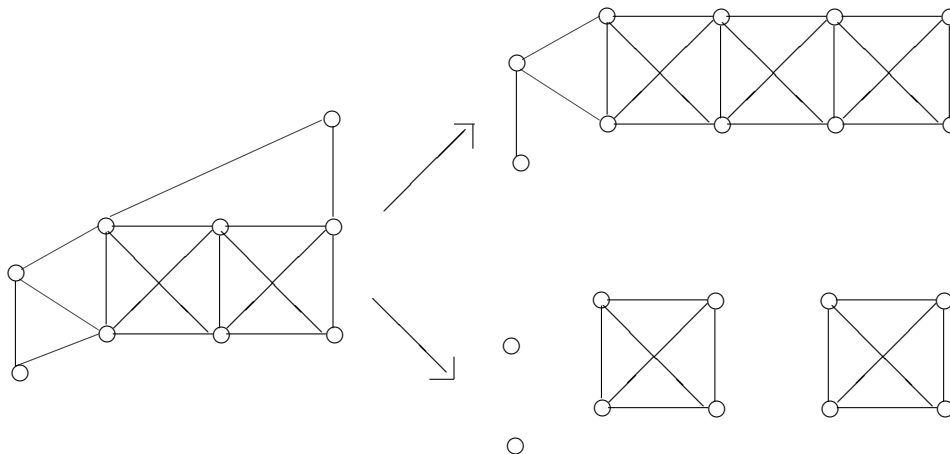


Figure 1: A graph and its 3-line graph and 3-anti-Gallai graph.

Definition 2. For an integer $k \geq 1$, the k -anti-Gallai graph $\Delta_k(G)$ of a graph G has one vertex for each k -clique of G , and two vertices in $\Delta_k(G)$ are adjacent if and only if the union of the two k -cliques represented by them span a $(k + 1)$ -clique in G .

Hence, $\Delta_k(G)$ is a subgraph of $L_k(G)$. For every graph G , its 1-anti-Gallai graph is G itself, whilst 2-anti-Gallai graph means anti-Gallai graph (denoted by $\Delta(G)$) in the usual sense. If a vertex c_i of $L_k(G)$ or $\Delta_k(G)$ represents the k -clique C_i of G , we say that c_i is the image of C_i and conversely, C_i is the preimage of c_i . In notation, if the context is clear, the preimage of c_i is denoted either by C_i or by $C(c_i)$. A graph G is called k -line graph or k -anti-Gallai graph if there exists a graph G' such that $L_k(G') = G$ or $\Delta_k(G') = G$ holds, respectively.

1.2 Results

The line graph operator is a classical subject in graph theory. From the rich literature here we mention only the forbidden subgraph characterization given by Beineke in 1970 [6]. The notion of the triangle graph and that of the k -line graph were introduced several times independently by different motivations, and studied from different points of view (see for example [5, 7, 8, 9, 12, 15, 16, 18]). For earlier results on anti-Gallai and k -anti-Gallai graphs we refer the reader to the papers [4, 10, 13] and the book [15]. As relates the most recent works, Anand *et al.* answered a question of Le by showing that the recognition problem of anti-Gallai graphs is NP-complete [2], moreover an application of the anti-Gallai graphs to automate the discovery of ambiguous words is described in [1].

In this paper we study three related topics. The first one concerns a question of Bagga [5] asking for a characterization of graphs G for which $G \square K_n$ is a triangle graph. As a complete solution in a much more general setting, in Section 2 we give a necessary and sufficient condition for a Cartesian product $G \square F$ to be a 3-line graph. Then, in Section 3, this result is generalized by establishing a unified characterization for $G \square F$ to be a k -line graph, for every $k \geq 2$.

We also study the algorithmic hardness of recognition problems. Due to the forbidden subgraph characterization in [6], the 2-line graphs can be recognized in polynomial time. In contrast to this, we prove in Section 4 that the analogous problem is NP-complete for the triangle graphs. Then, in Section 5 the same hardness is established for k -line graphs for each fixed $k \geq 4$. Via some lemmas and a constructive reduction, we obtain that recognizing k -anti-Gallai graphs is also NP-complete for each $k \geq 3$. The latter result solves a problem raised by Anand *et al.* [2], extending their theorem from $k = 2$ to larger values of k .

In Section 6, graphs with $L_k(G) \cong L(G)$ are identified for each $k \geq 3$. Finally, in the concluding section we put some remarks and formalize a problem which remains open.

1.3 Some basic facts

Here we list some basic statements, which can be found in [15] or can be proved directly from the definitions.

Observation 1 ([15]). *Every k -line graph is $K_{1,k+1}$ -free.*

Observation 2. *Every clique K_n of a k -line graph $L_k(G)$ either corresponds to n k -cliques of G sharing a fixed $(k-1)$ -clique, or corresponds to n k -cliques contained in a common K_{k+1} . In particular, every clique of order n in a triangle graph $\mathcal{T}(G)$ corresponds to n triangles of G which are either incident with a fixed edge, or contained in a common K_4 .*

Proof. Let c_1, \dots, c_n be the vertices of an n -clique K_n of $L_k(G)$ and C_1, \dots, C_n be the corresponding k -cliques in G . Moreover, let $v_1, \dots, v_k \in V(G)$ be the vertices which induce C_1 . Since c_2 is adjacent to c_1 in $L_k(G)$, the k -clique C_2 has precisely one vertex outside C_1 . We assume without loss of generality that $C_2 = \{u, v_2, v_3, \dots, v_k\}$. Now, suppose that there exists a vertex in K_n , say c_3 , such that its preimage C_3 does not contain some vertex from the set $C_1 \cap C_2 = \{v_2, v_3, \dots, v_k\}$; say, v_k is omitted. In this case, since c_3 is adjacent to both c_1 and c_2 , the k -clique C_3 must be induced by $\{u, v_1, v_2, \dots, v_{k-1}\}$. Then, for any further vertex c_i , the preimage must be of the form $C_i = \{u, v_1, v_2, \dots, v_k\} \setminus \{v_{j_i}\}$ for some $2 \leq j_i \leq k-1$. This proves that if not all the intersections $C_i \cap C_j$ are the same, then each of the k -cliques C_1, \dots, C_n is contained in the $(k+1)$ -clique $\{u, v_1, v_2, \dots, v_k\}$. \square

Observation 3. *If G is the k -line graph of a K_{k+1} -free graph, then for every $k' > k$, G is also the k' -line graph of a $K_{k'+1}$ -free graph.*

Proof. Let $G = L_k(H)$ for a K_{k+1} -free graph H . Consider the join $H' = H \vee K_{k'-k}$. Since H is K_{k+1} -free, H' is $K_{k'+1}$ -free and every k' -clique of H' originates from a k -clique of H extended by the $k' - k$ new vertices. Additionally, two k' -cliques of H' intersect in a $K_{k'-1}$ if and only if the corresponding k -cliques of H meet in a K_{k-1} . Consequently, $L_{k'}(H') = L_k(H) = G$. \square

2 Cartesian product and triangle graphs

In this section we solve a problem posed in [5] by Bagga.

Theorem 4. *The Cartesian product $G \square F$ of two nontrivial connected graphs is a triangle graph if and only if F is a complete graph and G is the line graph of a triangle-free graph (or vice versa).*

Before proving the theorem we verify a lemma.

Lemma 5. *If G contains a diamond as an induced subgraph then $G \square K_n$ is not a triangle graph for $n \geq 2$.*

Proof. To prove the lemma we apply the following result from [5].

- (*) If H is a triangle graph with $K_4 - e$ as an induced subgraph, then there exists a vertex x in H such that x is adjacent to three vertices of one triangle of $K_4 - e$ and nonadjacent to the fourth vertex.

Let G be a graph which contains a diamond induced by the vertices u_1, u_2, u_3 and u_4 , where (u_1, u_4) is the non-adjacent vertex pair. Assume for a contradiction that there exists a graph H whose triangle graph is $G \square K_n$ for some $n \geq 2$. Let $v_1 \in V(K_n)$. Then, $(u_1, v_1), (u_2, v_1), (u_3, v_1), (u_4, v_1)$ is an induced diamond in $G \square K_n$. Since $G \square K_n$ is a triangle graph, by (*), it must contain a vertex (u_5, v_1) which is adjacent to all vertices of one of the triangles in the diamond and not adjacent to the fourth vertex. Let (u_4, v_1) be the vertex which is not adjacent to (u_5, v_1) . Let t_i be the triangle in H corresponding to the vertex (u_i, v_1) in $G \square K_n$ for $i = 1, \dots, 5$. Then t_1, t_2, t_3 and t_5 must be the triangles of a K_4 and t_4 is a triangle which shares the edge which is common to the triangles t_2 and t_3 . Let $v_2 \in V(K_n) \setminus \{v_1\}$ (it exists, since $n \geq 2$). Then (u_1, v_2) is adjacent to (u_i, v_1) only for $i = 1$. Therefore, the triangle in H corresponding to the vertex (u_1, v_2) must share an edge with t_1 and not with any other t_i for $i = 2, \dots, 5$. But, each edge of t_1 is shared with at least one among t_2, t_3 and t_5 , which gives a contradiction. Therefore, $G \square K_n$ is not a triangle graph. \square

Proof of Theorem 4. If both G and F are non-complete graphs, then $G \square F$ contains an induced $K_{1,4} \subset P_3 \square P_3$ and hence, by Observation 1, it is not a triangle graph. So we can assume that $F = K_n$ for some $n \geq 2$.

If G is not a line graph, then by the theorem of Beineke [6], G contains one of the nine forbidden subgraphs as an induced subgraph (see Figure 2). If it is $K_{1,3}$, then $G \square K_n$ contains an induced $K_{1,4}$, which is forbidden for triangle graphs. In the case of any of the remaining eight graphs, G contains an induced diamond and hence, by Lemma 5, $G \square K_n$ cannot be a triangle graph.

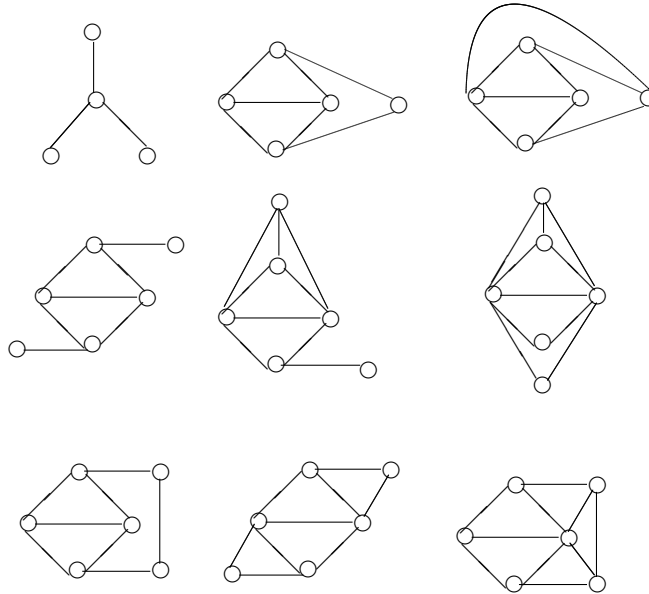


Figure 2: Forbidden subgraphs for line graph.

Let G be the line graph of a graph H which contains a triangle. Let $T = (u_1, u_2, u_3)$ be a triangle in H . If $H \neq K_3$, there exists a vertex u_4 adjacent to some u_i in T (not necessarily to u_i alone). But, then $G = L(H)$ contains a diamond and hence by Lemma 5, $G \square K_n$ is not a triangle graph. If H is K_3 itself, then $H = L(K_{1,3})$ also.

Conversely, let G be the line graph of a triangle-free graph H . Then $\mathcal{T}(H \vee \overline{K_n}) = G \square K_n$, completing the proof of the theorem. \square

Concerning edgeless graphs, note that $G \square K_1 \cong G$, and hence the characterization problem of graphs G such that $G \square K_1$ is a triangle graph is equivalent to characterizing triangle graphs.

3 Cartesian product and k -line graphs

As proved in Section 2, if both G and F are nontrivial connected graphs, $G \square F$ is a triangle graph if and only if one of G and F is a line graph of a K_3 -free graph and the other one is a complete graph. We will see that a direct analogue of this theorem is not valid for k -line graphs in general. For instance, one can observe that the grid graph $P_n \square P_m$ is a k -line graph for every n, m and $k \geq 4$.

Our main result in this section is the following theorem, which gives a necessary and sufficient condition for a product $G \square F$ of non-edgeless graphs to be a k -line graph. Recall that k -line graphs were defined for $k = 1$, too.

Theorem 6. *For every $k \geq 2$, the product $G \square F$ of two non-edgeless connected graphs is a k -line graph if and only if there exist positive integers k_1 and k_2 such that G is the k_1 -line graph of a K_{k_1+1} -free graph, F is the k_2 -line graph of a K_{k_2+1} -free graph and $k_1 + k_2 \leq k$ holds.*

If F is a complete graph then it is the k_2 -line graph of a K_{k_2+1} -free graph for every $k_2 \geq 1$. (For example, $K_n = L_{k_2}(K_{k_2-1} \vee nK_1)$ for all $k_2 \geq 1$, where the degenerate case $k_2 = 1$ simply means that $K_n = L_1(nK_1)$.) Hence, in this particular case of Theorem 6, the existence of an appropriate $k_1 \leq k-1$ is required. By Observation 3, this is equivalent to the claim that G is the $(k-1)$ -line graph of a K_k -free graph. Thus, we obtain:

Corollary 7. *For every two integers $n \geq 2$ and $k \geq 2$, the product $G \square K_n$ is a k -line graph if and only if G is the $(k-1)$ -line graph of a K_k -free graph.*

Theorem 6 will be proved at the end of this section. First we need some lemmas.

Lemma 8.

- (i) *If H contains a K_{k+1} subgraph, then for the corresponding $(k+1)$ -clique \mathcal{C} of $L_k(H)$, each vertex $c \in V(L_k(H)) \setminus \mathcal{C}$ is either adjacent to none of the vertices of \mathcal{C} or c is adjacent to exactly two vertices of \mathcal{C} .*
- (ii) *Assume that $k \geq 2$ and no component of the k -line graph $L_k(H)$ is isomorphic to K_{k+1} . Then, H is K_{k+1} -free if and only if $L_k(H)$ is diamond-free.*

Proof. First, assume that H contains a K_{k+1} , which is induced by the vertex set $V = \{v_1, \dots, v_{k+1}\} \subseteq V(H)$, and consider the corresponding $(k+1)$ -clique \mathcal{C} in $L_k(H)$ whose vertices c_1, \dots, c_{k+1} represent the k -subsets of V . If there is a further vertex c^* adjacent to at least one vertex of \mathcal{C} , then in H the k -clique C^* , which is the preimage of c^* , intersects V in exactly $k-1$ vertices. Without loss of generality, we assume that for every c_i the preimage is $C_i = V \setminus \{v_i\}$, moreover that $C^* \cap V = \{v_1, \dots, v_{k-1}\}$. Then, $|C^* \cap C_k| = |C^* \cap C_{k+1}| = k-1$, but $|C^* \cap C_i| = k-2$ holds for every $1 \leq i \leq k-1$. Therefore, c^* is adjacent to exactly two vertices of \mathcal{C} . This verifies (i). Concerning (ii), note that if the $(k+1)$ -clique \mathcal{C} is not a component of $L_k(H)$ then such a c^* surely exists, and if $k \geq 2$, vertices c_1, c_k, c_{k+1}, c^* induce a diamond in $L_k(H)$.

For the other direction of (ii), assume that H is K_{k+1} -free, and in $L_k(H)$ vertices c_1, c_2, c_3, c_4 induce a diamond where c_2 and c_4 are nonadjacent. By Observation 2, the preimages C_1, C_2, C_3 of c_1, c_2, c_3 are three different k -cliques of H sharing a fixed $(k-1)$ -clique. Then, $|C_1 \cap C_2 \cap C_3| = k-1$ and similarly, $|C_1 \cap C_3 \cap C_4| = k-1$ hold. By the adjacency of c_1 and c_3 , $|C_1 \cap C_3| = k-1$ is valid as well. Hence, the vertex sets $C_1 \cap C_2 \cap C_3$ and $C_1 \cap C_3 \cap C_4$ must be the same, contradicting the non-adjacency of c_2 and c_4 . Thus, we conclude that for a K_{k+1} -free H , the k -line graph contains no induced diamond. \square

Lemma 9. *If the Cartesian product $G \square F$ of two non-edgeless graphs is the k -line graph of H , then H is K_{k+1} -free.*

Proof. Suppose for a contradiction that H contains a complete subgraph X induced by vertices x_1, \dots, x_{k+1} . Then, in $L_k(H) = G \square F$ the vertices c_1, \dots, c_{k+1} , representing the k -cliques $C_1, \dots, C_{k+1} \subset X$, form a complete subgraph. Hence, all these vertices

c_1, \dots, c_{k+1} must belong either to the same copy of G or to the same copy of F . Assume without loss of generality that $c_i = (v_1, u_i)$ for every $1 \leq i \leq k+1$, and let v_j be a neighbor of v_1 in G . Then, the vertex (v_j, u_1) is adjacent to only one vertex (namely, to (v_1, u_1)) from the $(k+1)$ -clique. This contradicts Lemma 8(i) and hence, H is K_{k+1} -free. \square

Lemma 10. *If G is the k -line graph of H , and G contains an induced cycle $c_1c_2c_3c_4$, then the corresponding k -cliques C_1, C_2, C_3, C_4 of H satisfy $C_2 \setminus C_1 = C_3 \setminus C_4$.*

Proof. As the 1-line graph $L_1(H)$ is a complete graph, it contains no induced four-cycle. Hence, we can assume that $k \geq 2$. Let $C_1 \setminus C_2 = \{v_1\}$, $C_2 \setminus C_1 = \{v_2\}$, $C_1 \setminus C_4 = \{z_1\}$ and $C_4 \setminus C_1 = \{z_2\}$. We observe that $v_1 \neq z_1$ and $v_2 \neq z_2$, as any $v_i = z_i$ would mean the adjacency of c_2 and c_4 . Now, suppose for a contradiction that $v_2 \notin C_3$. Since c_1c_2 and c_2c_3 are edges in G and $v_2 \in C_2$, but $v_2 \notin C_1$ and $v_2 \in C_3$, it follows that, $C_3 \cap C_2 = C_2 \setminus \{v_2\} = C_1 \cap C_2$. This implies $|C_1 \cap C_3| = k - 1$, which contradicts $c_1c_3 \notin E(G)$. Therefore, $v_2 \in C_3$ holds and since $v_2 \notin C_4$, the desired equality $C_3 \setminus C_4 = \{v_2\} = C_2 \setminus C_1$ follows. \square

In view of Lemma 10, we can give a structural characterization for graphs whose k -line graph is the Cartesian product $G \square F$ of two non-edgeless graphs. We will use the following notions.

If $L_k(H) = G \square F$, consider a copy G_i and the k -cliques of H which are represented by the vertices of G_i . A vertex contained in all these k -cliques is called a *universal vertex*, otherwise it is *non-universal* (with respect to G_i). Formally, let

$$U_i^G(H) = \bigcap \{C(v_j, u_i) : v_j \in V(G)\}, \text{ and } X_i^G(H) = \bigcup \{C(v_j, u_i) : v_j \in V(G)\} \setminus U_i^G(H).$$

Analogously, the sets U_i^F and X_i^F of universal and non-universal vertices with respect to the copy F_i are also introduced.

Lemma 11. *Let G and F be two connected graphs and let $L_k(H) = G \square F$.*

- (i) *For every two copies G_i and G_j of G the non-universal vertices are the same: $X_i^G(H) = X_j^G(H)$.*
- (ii) *Each copy G_i is the k -line graph of the subgraph induced by $U_i^G(H) \cup X_i^G(H)$.*

Proof. First assume that u_i and u_j are adjacent vertices in F . For every two vertices v_a and v_b of G there is a path between (v_a, u_i) and (v_b, u_i) in G_i . This path together with the corresponding vertices of G_j induces a chain of 4-cycles. Hence, the repeated application of Lemma 10 implies that

$$C(v_a, u_i) \setminus C(v_a, u_j) = C(v_b, u_i) \setminus C(v_b, u_j) \quad \text{and} \quad C(v_a, u_j) \setminus C(v_a, u_i) = C(v_b, u_j) \setminus C(v_b, u_i)$$

for every two vertices v_a and v_b of G . As follows, there are fixed vertices $w_{i,j} \in U_i^G(H)$ and $w_{j,i} \in U_j^G(H)$ such that the preimage of any vertex of copy G_j can be obtained from

the preimage of the corresponding vertex of copy G_i by replacing $w_{i,j}$ with $w_{j,i}$ in the k -clique. Thus, $U_j^G(H) = (U_i^G(H) \setminus \{w_{i,j}\}) \cup \{w_{j,i}\}$ and for the non-universal vertices $X_i^G(H) = X_j^G(H)$ holds. Since F is connected, the latter equality is valid for nonadjacent vertices $u_i, u_j \in V(F)$ as well. This verifies (i).

To prove (ii), observe that $U_i^G(H) \neq U_j^G(H)$ and $|U_i^G(H)| = |U_j^G(H)|$ for every pair i, j . Therefore, neither $U_i^G(H) \setminus U_j^G(H)$ nor $U_j^G(H) \setminus U_i^G(H)$ is empty. Consequently, every k -clique in the subgraph induced by $U_i^G(H) \cup X_i^G(H)$ has its image in copy G_i . This proves (ii). \square

Proof of Theorem 6. To prove sufficiency, let $G = L_{k_1}(G')$ and $F = L_{k_2}(F')$ where G' is K_{k_1+1} -free, F' is K_{k_2+1} -free, and $k_1 + k_2 \leq k$. Then, the join $H' = G' \vee F'$ is $K_{k_1+k_2+1}$ -free and every $(k_1 + k_2)$ -clique of H' originates from a k_1 -clique of G' and from a k_2 -clique of F' . Thus, the vertices of $L_{k_1+k_2}(H')$ correspond to the pairs (v_i, u_j) with $v_i \in V(G')$ and $u_j \in V(F')$. Moreover two vertices (v_i, u_j) and (v_k, u_ℓ) in $L_{k_1+k_2}(H')$ are adjacent if and only if

- either $v_i = v_k$ and the k_2 -cliques $C(u_j)$ and $C(u_\ell)$ share a $(k_2 - 1)$ -clique in F' , that is $u_j u_\ell \in E(F)$,
- or $u_j = u_\ell$ and the k_1 -cliques $C(v_i)$ and $C(v_k)$ share a $(k_1 - 1)$ -clique in G' , that is $v_i v_k \in E(G)$.

Therefore, $L_{k_1+k_2}(H') = G \square F$, and by Observation 3, $L_k(H' \vee K_{k-k_1-k_2}) = G \square F$ also holds for every $k > k_1 + k_2$.

To prove necessity, suppose that $H = G \square F$ is the k -line graph of H' . By Lemma 9, H' is K_{k+1} -free. Consider the sets $A = U_1^G(H')$ and $B = U_1^F(H')$ of universal vertices in copies G_1 and F_1 .

Claim A. G is the $(k - |A|)$ -line graph of a $K_{k-|A|+1}$ -free graph and F is the $(k - |B|)$ -line graph of a $K_{k-|B|+1}$ -free graph.

Proof. By Lemma 11(ii), $G_1 \cong G$ is the k -line graph of the subgraph $G^* \subset H'$ induced by $A \cup X_1^G(H')$. If the universal vertices of G^* are deleted, we obtain $G' = G^* - A$. While constructing G' from G^* , every k -clique is shrunk into a $(k - |A|)$ -clique and two cliques share exactly $k - |A| - 1$ vertices if and only if the corresponding vertices of G are adjacent. This proves that $G = L_{k-|A|}(G')$. It is clear that G' is $K_{k-|A|+1}$ -free. The analogous argument for F yields $F' = F^* - B$ such that $F = L_{k-|B|}(F')$ and F' is $K_{k-|B|+1}$ -free.

Claim B. $|A| + |B| \geq k$.

Proof. Assume to the contrary that $|A| + |B| < k$. Then, there exists a vertex $z \in C(v_1, u_1) \setminus (A \cup B)$. As the graph $G_1 \cong G$ is connected and z cannot be contained in all preimages $C(v_\ell, u_1)$, there exist adjacent vertices (v_i, u_1) and (v_j, u_1) such that $z \in C(v_i, u_1)$ and $z \notin C(v_j, u_1)$. This means $C(v_i, u_1) \setminus C(v_j, u_1) = \{z\}$. Similarly for F_1 , there exist indices m and n such that $C(v_1, u_m) \setminus C(v_1, u_n) = \{z\}$ holds. By Lemma

10, for the 4-cycle induced by $\{(v_i, u_n), (v_i, u_m), (v_j, u_m), (v_j, u_n)\}$ the following equalities hold:

$$\begin{aligned} C(v_1, u_m) \setminus C(v_1, u_n) &= C(v_i, u_m) \setminus C(v_i, u_n) = \{z\}, \\ C(v_i, u_1) \setminus C(v_j, u_1) &= C(v_i, u_n) \setminus C(v_j, u_n) = \{z\}. \end{aligned}$$

They give a contradiction on the question whether z is in $C(v_i, u_n)$ or not. This proves $|A| + |B| \geq k$.

Denoting $k_1 = k - |A|$ and $k_2 = k - |B|$, $k_1 + k_2 \leq k$ follows by Claim B. Then, Claim A proves the necessity of the condition given for G and F in the theorem. \square

The proof of Theorem 6 also verifies the following statement.

Corollary 12. *If G is K_{k_1+1} -free and F is K_{k_2+1} -free, then*

$$L_{k_1}(G) \square L_{k_2}(F) \cong L_{k_1+k_2}(G \vee F).$$

4 NP-completeness of recognizing triangle graphs

As is well-known, the line graphs can be recognized in polynomial time due to the forbidden subgraph characterization by Beineke [6]. Also linear-time algorithms were designed for solving this problem [14, 17]. Here we prove that triangle graphs (that is, 3-line graphs) are hard to recognize.

Theorem 13. *The following problems are NP-complete:*

- (i) *Recognizing triangle graphs.*
- (ii) *Deciding whether a given graph is the triangle graph of a K_4 -free graph.*

Moreover, both problems remain NP-complete on the class of connected graphs.

Before proving Theorem 13, we verify two lemmas which give necessary conditions for graphs to be anti-Gallai or triangle graphs of some K_4 -free graph, respectively.

Lemma 14. *Assume that F is a connected non-trivial graph and F is the anti-Gallai graph of F' . Then F' is K_4 -free if and only if every edge of F is contained in exactly one triangle, or equivalently*

- (\star) *every maximal clique of F is a triangle and any two triangles share at most one vertex.*

Proof. If F' contains a K_4 subgraph then $F = \Delta(F')$ contains an induced $K_6 - 3K_2$ and (\star) does not hold. If F' is K_4 -free, any three pairwise adjacent vertices in $\Delta(F')$ correspond to three edges of F' which form a triangle. Hence no edge of $\Delta(F')$ belongs to more than one triangle. Additionally, by definition, in an anti-Gallai graph every edge corresponds to two preimage-edges of a triangle; hence, every edge of $\Delta(F')$ is contained in a K_3 . Since F is assumed to be connected and non-edgeless, it contains no isolated vertices. This proves that every maximal clique of F is a K_3 and any two triangles have at most one vertex in common. \square

Lemma 15. Assume that G is a connected graph which is not isomorphic to K_4 , moreover $G = \mathcal{T}(G')$. Then G' is K_4 -free if and only if

($\star\star$) each vertex of G is contained in at most three maximal cliques and these cliques are pairwise edge-disjoint.

Proof. In this proof, the vertex of G whose preimage is a triangle abc in G' will be denoted by t_{abc} .

First suppose that G' contains a K_4 induced by the vertices x, y, z, u . Clearly, the four triangles of the K_4 correspond to a 4-clique in $\mathcal{T}(G')$. By our condition $\mathcal{T}(G')$ is not a 4-clique, hence there exists a triangle in G' , containing exactly two vertices from x, y, z, u . Say, this triangle is xyw . In $\mathcal{T}(G')$, the vertex originated from xyz is contained in both cliques induced by the vertex sets $\{t_{xyz}, t_{xyu}, t_{yzu}, t_{xzu}\}$ and $\{t_{xyz}, t_{xyu}, t_{xyw}\}$, respectively. Maybe the second clique is not maximal, but since there is no edge between t_{yzu} and t_{xyw} , there are two different maximal cliques with the common edge $t_{xyz}t_{xyu}$. This shows that ($\star\star$) does not hold.

For the converse, suppose that G' is K_4 -free. Then by Observation 2, each clique of $\mathcal{T}(G')$ corresponds to triangles sharing a fixed edge in G' . Thus, a vertex $t_{xyz} \in V(\mathcal{T}(G))$ can be contained only in those maximal cliques which correspond to the three edges of its preimage-triangle (one or two of these cliques might be missing) and any two of these maximal cliques have t_{xyz} as the only common vertex, hence ($\star\star$) holds. \square

While proving that the recognition problem of triangle graphs is NP-complete, we will use the following notion. The *triangle-restriction* of a graph is obtained if the edges not contained in any triangles and the possibly arising isolated vertices are deleted. Every graph has a triangle-restriction, and the application of this operator changes neither the anti-Gallai graph (if it is connected)¹, nor the triangle graph. A graph is called *triangle-restricted* if each edge and each vertex of it belongs to a triangle. The *clique graph* $\mathcal{K}(G)$ of a graph G is the intersection graph of the set of all *maximal* cliques of G .²

Proof of Theorem 13. The decision problems are clearly in NP. The NP-completeness of (ii) will be reduced from the following theorem recently proved by Anand et al. [2]: Deciding whether a connected graph F is the anti-Gallai graph of some K_4 -free graph is an NP-complete problem.

Consider an instance F to decide whether it is the anti-Gallai graph of a K_4 -free graph. In the first step, we check the necessary condition (\star); if it does not hold, F is not the anti-Gallai graph of any K_4 -free graphs. From now on, suppose that (\star) holds for F . Then every maximal clique of F is a triangle and the clique graph $G = \mathcal{K}(F)$ is exactly the triangle-intersection graph of F . If F is connected then so is G , and $G \cong K_4$ holds if and only if F is the union of four triangles sharing exactly one vertex. Hence,

¹If some edges of a graph F' are not contained in any triangles, their images in the anti-Gallai graph are isolated vertices. The deletion of these edges from F' results in the deletion of all isolated vertices from the anti-Gallai graph.

²That is, the vertices of $\mathcal{K}(G)$ correspond to the maximal cliques of G and two vertices of $\mathcal{K}(G)$ are adjacent if the corresponding cliques share at least one vertex.

from now on we assume that $G \not\cong K_4$. In addition, if F fulfills property (\star) , then its triangle-intersection graph G fulfills property $(\star\star)$.

Next, we prove that F is the anti-Gallai graph of a K_4 -free triangle-restricted graph H if and only if G is the triangle graph of H .

Assume that $F = \Delta(F')$. By (\star) , F' is K_4 -free, hence its triangles are in one-to-one correspondence with the triangles of F and by (\star) this yields a one-to-one correspondence with the vertices of G . Moreover two triangles in F' share an edge if and only if the corresponding triangles share a vertex in F ; and if and only if the corresponding vertices in the clique graph G are adjacent. Therefore, $G = \mathcal{T}(F')$.

To prove the other direction, assume that $G = \mathcal{T}(G')$. Since G satisfies $(\star\star)$, G' must be K_4 -free. We can choose G' to be triangle-restricted. Now, for every vertex $t \in V(G)$, if t is contained in only two maximal cliques, then in addition the 1-element vertex set $\{t\}$ will also be considered as a ‘maximal clique’ of G . Similarly, if t is contained in only one clique of G , then $\{t\}$ is also taken as a ‘maximal clique’ with multiplicity 2. Then the edges of G' are in one-to-one correspondence with the maximal cliques of G . These maximal cliques are in one-to-one correspondence with the vertices of F , where the one-element cliques of G represent vertices contained in only one triangle of F . Also, two edges of G' belong to a common triangle if and only if the corresponding maximal cliques have a common vertex (which represents the triangle); that is, if and only if the two vertices of F , represented by the cliques, are adjacent. This proves $F = \Delta(G')$.

Checking (\star) and constructing G from F takes polynomial time. So, the recognition problem of the anti-Gallai graphs can be reduced to that of the triangle graphs in polynomial time. Hence, the recognition problem of triangle graphs is NP-complete and this remains valid on the class of connected graphs satisfying $(\star\star)$. \square

5 Recognizing generalized line graphs and anti-Gallai graphs

In this section we turn to the recognition problems of general k -line graphs and k -anti-Gallai graphs. In sharp contrast to the linear-time recognizability of k -line graphs for $k \leq 2$, by Theorem 13 the analogous problem is NP-complete for $k = 3$. Also, anti-Gallai graphs are hard to recognize as proved by Anand et al. via a reduction from 3-SAT [2]. Now, we complete these results by proving that the recognition problems of k -line graphs and k -anti-Gallai graphs are NP-complete for each $k \geq 3$.

Theorem 16. *The following problems are NP-complete for every fixed $k \geq 3$:*

(i) *Recognizing k -line graphs.*

(ii) *Deciding whether a given graph is the k -line graph of a K_{k+1} -free graph.*

Moreover both problems remain NP-complete on the class of connected graphs.

Proof. Clearly, problems (i) and (ii) are in NP. Moreover, by Theorem 13, both problems are NP-complete for $k = 3$, already on the class of connected graphs. Therefore, we can proceed by induction on k .

For the inductive step, assume that $k \geq 4$ and that (ii) is NP-complete for $k - 1$ on the class of connected graphs. Let G be a connected graph and construct the Cartesian product $H = G \square K_2$, which is also connected. Due to Corollary 7, G is a $(k - 1)$ -line graph of a K_k -free graph if and only if H is a k -line graph of a K_{k+1} -free graph. Therefore, (ii) is NP-complete for every $k \geq 3$. On the other hand, by Lemma 9 a graph of the form $G \square K_2$ is a k -line graph if and only if it is a k -line graph of a K_{k+1} -free graph. Hence, the above reduction also proves the NP-completeness of (i) for every $k \geq 3$. \square

Before proving the same hardness for the recognition problem of k -anti-Gallai graphs, we state a lemma. Note that part (i) gives the same condition (namely diamond-freeness) for $\Delta_k(G)$ as Lemma 8 does for $L_{k+1}(G)$ to ensure that G is K_{k+2} -free.

Lemma 17. *For every $k \geq 2$, every graph G and its k -anti-Gallai graph $\Delta_k(G)$ satisfy the following relations:*

- (i) G is K_{k+2} -free if and only if $\Delta_k(G)$ is diamond-free.
- (ii) G is K_{k+2} -free if and only if each maximal clique of $\Delta_k(G)$ is either an isolated vertex or a $(k + 1)$ -clique. Moreover any two maximal cliques intersect in at most one vertex.

Proof. First, assume that G contains a $(k + 2)$ -clique induced by the vertex set $V = \{v_1, \dots, v_{k+2}\}$. Consider the following k -cliques:

$$C_1 = V \setminus \{v_1, v_2\}, \quad C_2 = V \setminus \{v_2, v_3\}, \quad C_3 = V \setminus \{v_1, v_3\}, \quad C_4 = V \setminus \{v_1, v_4\}.$$

Any two of these k -cliques are in a common $(k + 1)$ -clique except the pair (C_2, C_4) . Therefore, in the k -anti-Gallai graph the corresponding vertices c_1, c_2, c_3 and c_4 induce a diamond. In addition, the two maximal cliques containing $c_1 c_2 c_3$ and $c_1 c_3 c_4$, respectively, must be different and intersect in more than one vertex.

To prove the other direction of (i) and (ii), assume that G is K_{k+2} -free. First, consider an edge $c_i c_j \in E(\Delta_k(G))$. The union $C_i \cup C_j$ of the represented k -cliques induces a K_{k+1} subgraph in G , whose k -clique subgraphs are represented by vertices forming a K_{k+1} subgraph in $\Delta_k(G)$. Hence, every edge of $\Delta_k(G)$ belongs to a $(k + 1)$ -clique. Now, suppose that $c_i c_j c_\ell$ is a triangle in $\Delta_k(G)$. The union $C_i \cup C_j$ of the preimage cliques induces a K_{k+1} in G . Also $C_i \cup C_j \cup C_\ell$ induces a complete subgraph as every two of its vertices are contained in a common clique. Since G is K_{k+2} -free, $C_i \cup C_j \cup C_\ell$ is a $(k + 1)$ -clique as well, and $C_\ell \subset C_i \cup C_j$ must hold. This implies that in the k -anti-Gallai graph, every two adjacent vertices c_i, c_j together with all their common neighbors form a $(k + 1)$ -clique. As follows concerning (i), $\Delta_k(G)$ contains no induced diamond. Furthermore, each edge belongs to exactly one maximal clique of $\Delta_k(G)$ and this must be a $(k + 1)$ -clique. These complete the proof of (i) and (ii). \square

Theorem 18. *The following problems are NP-complete for every fixed $k \geq 3$:*

- (i) *Recognizing k -anti-Gallai graphs.*

- (ii) *Recognizing k -anti-Gallai graphs on the class of connected and diamond-free graphs.*
- (iii) *Deciding whether a given connected graph is a k -anti-Gallai graph of a K_{k+2} -free graph.*

Proof. The membership in NP is obvious for each of (i)–(iii). As diamond-freeness can be checked in polynomial time, statements (ii) and (iii) imply each other by Lemma 17(i). It is also clear that (ii) implies (i). Then, it is enough to prove (iii). For $k = 2$, problem (iii) was proved to be NP-complete in [2].

We proceed by induction on k . Consider a generic connected instance G and an integer $k \geq 3$.

For each fixed k , the condition given in Lemma 17(ii) can be checked in polynomial time. If it does not hold, G cannot be a k -anti-Gallai graph of any K_{k+2} -free graph. From now on we suppose that every maximal clique of G is of order $k + 1$ and any two maximal cliques have at most one vertex in common. For such a G we construct the following graph G_e and prove that G is a k -anti-Gallai graph if and only if G_e is a $(k + 1)$ -anti-Gallai graph.

Construction of G_e . Take two disjoint copies G^1 and G^2 of G with vertex sets $V(G^j) = \{c_i^j : c_i \in V(G)\}$ ($j = 1, 2$), moreover one vertex b_s for each $(k + 1)$ -clique B_s of G . Besides the edges of G^1 and G^2 take all edges of the form $b_s c_i^j$ for which $c_i \in B_s$ and $j = 1, 2$ hold.

As G consists of $(k + 1)$ -cliques such that any two of them intersect in at most one vertex, G_e consists of $(k + 2)$ -cliques such that any two of them intersect in at most one vertex.

Claim C. If $G = \Delta_k(G')$ then $G_e = \Delta_{k+1}(G' \vee 2K_1)$.

Proof. Let \mathcal{B} denote the set of k -cliques of G' . Corresponding to the relation $G = \Delta_k(G')$, we have a bijection $\phi : \mathcal{B} \mapsto V(G)$ such that every k -clique of G' is mapped to the vertex representing it in $\Delta_k(G')$.

Partition the set \mathcal{A} of $(k + 1)$ -cliques of the join $G'_e = G' \vee \{z_1, z_2\}$ into three subsets. An $A \in \mathcal{A}$ is said to be of Type 1 or 2 or 3 if it contains z_1 , or z_2 , or none of them, respectively. (Since z_1 and z_2 are nonadjacent, no clique contains both of them.)

Now, define a bijection $\varphi : \mathcal{A} \mapsto V(G_e)$ as follows. For every $A \in \mathcal{A}$,

$$\varphi(A) = \begin{cases} (\phi(A \setminus \{z_1\}))^1 & \text{if } A \text{ is of Type 1,} \\ (\phi(A \setminus \{z_2\}))^2 & \text{if } A \text{ is of Type 2,} \\ b_\ell & \text{if } A \text{ is of Type 3, and } A \text{ is the } (k + 1)\text{-clique } B_\ell \text{ of } G'. \end{cases}$$

To prove Claim C, we show that two $(k + 1)$ -cliques A_1 and A_2 of G'_e are contained in a common K_{k+2} if and only if $\varphi(A_1)$ and $\varphi(A_2)$ are adjacent in G_e .

- Type-1 cliques are mapped onto $V(G^1)$. In addition, two cliques A_1, A_2 of Type 1 are contained in a common $(k + 2)$ -clique in G'_e if and only if $A_1 \setminus \{z_1\}$ and $A_2 \setminus \{z_1\}$ are in a common $(k + 1)$ -clique in G' ; or equivalently, if and only if $(\phi(A_1 \setminus \{z_1\}))^1$ and $(\phi(A_2 \setminus \{z_1\}))^1$ are adjacent in G^1 . Similarly, Type-2 cliques are mapped onto $V(G^2)$ and the adjacencies in G^2 correspond to the adjacencies required in $\Delta_{k+1}(G'_e)$.

- If A_1 is of Type 1 and A_2 is of Type 3, their images are adjacent in $\Delta_{k+1}(G'_e)$ if and only if $A_1 \setminus \{z_1\} \subset A_2$; that is, if the $(k+1)$ -clique A_2 contains the k -clique $A_1 \setminus \{z_1\}$ in G' . This corresponds to the adjacency defined in Construction of G_e . The analogous property holds for cliques of Type 2 and Type 3.
- Since z_1 and z_2 are nonadjacent, no two cliques, one of Type 1 and the other of Type 2, belong to a common K_{k+2} in G'_e . Correspondingly, by the construction, there is no edge between $V(G^1)$ and $V(G^2)$ in G_e . Finally, as G' is K_{k+2} -free, no two $(k+1)$ -cliques of Type 3 are in a common $(k+2)$ -clique in G'_e . This corresponds to the fact that $V(G_e) \setminus (V(G^1) \cup V(G^2))$ is an independent vertex set.

These observations prove that $G_e = \Delta_{k+1}(G'_e)$.

Concerning the following claim, let us recall that Construction of G_e is applied for a $(K_{k+2}, \text{diamond})$ -free graph G , and yields a $(K_{k+3}, \text{diamond})$ -free G_e .

Claim D. If $G_e = \Delta_{k+1}(F')$ then there exist two vertices $z_1, z_2 \in V(F')$ such that $G = \Delta_k(F' - \{z_1, z_2\})$.

Proof. By Lemma 17, F' must be K_{k+3} -free. Consider a $(k+2)$ -clique D_ℓ of G_e . This contains exactly one vertex from $V(G_e) \setminus (V(G^1) \cup V(G^2))$, say b_ℓ , and assume that the other vertices of D_ℓ are from $V(G^1)$. The preimages of the vertices of D_ℓ are exactly the $(k+1)$ -clique subgraphs of a $(k+2)$ -clique A_ℓ of F' . There is a unique vertex u_ℓ , called complementing vertex of D_ℓ , such that $u_\ell \in A_\ell$. Moreover it is not contained in the preimage $C(b_\ell)$ but is contained in the preimage of each further vertex of D_ℓ . For this vertex, $A_\ell \setminus C(b_\ell) = \{u_\ell\}$ holds, and $C(c_i^1) \setminus C(b_\ell) = \{u_\ell\}$ is valid for every $c_i^1 \in D_\ell$.

Assume for a contradiction that there exist two different complementing vertices for the $(k+2)$ -cliques meeting $V(G^1)$. By the connectivity of G^1 , there exist two $(k+2)$ -cliques, say D_1 and D_2 , intersecting in a vertex c_i^1 with complementing vertices $u_1 \neq u_2$. Then, consider the vertices $b_1 \in D_1$, $b_2 \in D_2$, the induced 4-cycle $c_i^1 b_1 c_i^2 b_2$ in G_e and the preimage $(k+1)$ -cliques C_i^1 , B_1 , C_i^2 , B_2 . Since $c_i^1 b_1, c_i^1 b_2 \in E(G_e)$, there exist vertices x and y in F' such that

$$B_1 = C_i^1 \setminus \{u_1\} \cup \{x\}, \quad B_2 = C_i^1 \setminus \{u_2\} \cup \{y\}.$$

By our assumption, $u_1 \neq u_2$. Moreover, $x \neq y$ must be valid, since $x = y$ would imply for $B_1 \cup B_2 = C_i^1 \cup \{x\}$ to be a $(k+2)$ -clique, contradicting $b_1 b_2 \notin E(G_e)$. Further, if any two vertices coincide from the remaining pairs of u_1, u_2, x, y , it would contradict the above definition of x and y . Hence, u_1, u_2, x, y are four different vertices and the intersection $M = C_i^1 \cap B_1 \cap B_2 = B_1 \cap B_2$ is a $(k-1)$ -clique. Observe that the vertices in $M \cup \{u_1, u_2, x, y\}$ are pairwise adjacent as contained together in at least one of the $(k+2)$ -cliques $C_i^1 \cup B_1$ and $C_i^1 \cup B_2$, the only exception is the pair x, y . They are surely nonadjacent, as otherwise $M \cup \{u_1, u_2, x, y\}$ would be a forbidden $(k+3)$ -clique in F' .

Next, consider the k -element intersections $C_i^2 \cap B_1$ and $C_i^2 \cap B_2$. Both of them must contain the entire M and one further vertex from $\{x, u_2\}$ and $\{y, u_1\}$, respectively. But all the four possible choices are forbidden. The choice (u_2, u_1) would mean $C_i^1 = C_i^2$; any

of the choices (x, u_1) or (u_2, y) would imply that $C_i^1 \cup C_i^2$ is a $(k+2)$ -clique, contradicting $c_i^1 c_i^2 \notin E(G_e)$; and finally, the choice (x, y) contradicts the non-adjacency of x and y . By this contradiction we conclude that all $(k+2)$ -cliques intersecting G^1 have the same complementing vertex, say z_1 , and the similar result for G^2 with the common complementing vertex z_2 also follows. Deleting these vertices from F' , we obtain the graph F'' for which $\Delta_k(F'') = G$ holds.

Via Claims C and D, we have proved that G is the k -anti-Gallai graph of some K_{k+2} -free graph if and only if G_e is the $(k+1)$ -anti-Gallai graph of some K_{k+3} -free graph. We conclude that if problem (iii) is NP-complete for an integer $k \geq 2$, the same hardness follows for $k+1$. Moreover, the reduction takes time polynomial in terms of $|V(G)|$ and k . This proves (iii) from which (i) and (ii) directly follow. \square

6 Graphs with isomorphic line graph and k -line graph

Lemma 19. *If K_n is a subgraph of $L_k(G)$ for $n \geq k+2$, then the k -cliques in G corresponding to these n vertices in $L_k(G)$ share $k-1$ vertices.*

Proof. By Observation 2, the vertices of K_n either correspond to n k -cliques of G contained in a common K_{k+1} , or correspond to n k -cliques sharing a fixed $(k-1)$ -clique. The former case is impossible if $n \geq k+2$. Hence, the statement follows. \square

Theorem 20. *Let G be a connected graph. Then, for any $k \geq 3$, $L_k(G) \cong L(G)$ holds if and only if $G = K_{k+2}$.*

Proof. We begin with the remark that K_{k+2} indeed satisfies $L_k(K_{k+2}) \cong L(K_{k+2})$. Isomorphism can be established by the vertex-complementarity of edges (2-cliques) and k -cliques. Two edges of K_{k+2} share a vertex (and hence are adjacent in $L(K_{k+2})$) if and only if their complementing k -tuples (k -cliques) share exactly $k-1$ vertices (and hence are adjacent in $L_k(K_{k+2})$).

The rest of the proof is devoted to the “only if” part. Let G be a connected graph such that $L_k(G) \cong L(G)$. Let $t = \omega(L_k(G)) = \omega(L(G))$. If $t \geq k+2$, then by Lemma 19, the k -cliques in G corresponding to the t vertices in $L_k(G)$ that induce a t -clique, share $k-1$ vertices.

Therefore, $\Delta(G) \geq k-2+t$ and hence $\omega(L(G)) \geq k-2+t > t$ for $k \geq 3$, which contradicts $\omega(L(G)) = t$.

Therefore, $t \leq k+1$, which means $\omega(L(G)) \leq k+1$. Thus $\Delta(G) \leq k+1$, from which $\omega(G) \leq k+2$ clearly follows. Moreover, since, $L_k(G)$ is not the null graph, $\omega(G) \geq k$ also holds. Hence, we have $k \leq \omega(G) \leq k+2$. In the rest of the proof we consider the three possible values of ω .

Case 1. $\omega(G) = k$

We saw earlier that a 4-clique in $L_k(G)$ would require in G either a $(k+1)$ -clique or a $(k-1)$ -clique with four external neighbors. Hence in the current situation with $\Delta(G) \leq k+1$ and $\omega(G) = k$ we must have $\omega(L_k(G)) \leq 3$, which also means $\omega(L(G)) \leq 3$

and therefore G has $\Delta(G) \leq 3$. Then $\omega(G) \leq 4$, that is $k = 3$ or 4 . We show that these cases cannot occur.

For $k = 3$, the condition $\Delta(G) \leq 3$ implies that $\Delta(L_3(G)) \leq 1 < 2 \leq \Delta(L(G))$ holds, hence $L_3(G) \not\cong L(G)$. For $k = 4$, $\Delta(G) \leq 3$ yields $G \cong K_4$, thus $L_4(K_4) = K_1 \not\cong L(K_4)$. Consequently, $L_k(G) \not\cong L(G)$ if $\omega(G) = k$.

Case 2. $\omega(G) = k + 1$

Subcase A: G contains only one K_{k+1}

Since $\Delta(G) \leq k + 1$, G contains no $(k - 1)$ -clique with $k + 1$ common external neighbors. Hence, $L_k(G)$ also has only one K_{k+1} . Therefore, $L(G)$ also contains only one K_{k+1} and hence G has only one vertex of degree $k + 1$. It means that only one vertex of K_{k+1} in G has a neighbor outside the K_{k+1} , so that $L_k(G) = L_k(K_{k+1}) = K_{k+1}$ (for G is connected, $L(G)$ is connected and hence $L_k(G)$ is connected). But, $L(G) \neq K_{k+1}$. Therefore, G must contain more than one K_{k+1} .

Subcase B: G contains two K_{k+1} subgraphs which share $l > 0$ vertices

In this case, $\Delta(G) \geq 2k - l + 1$. By $\Delta(G) \leq k + 1$ we see that $l = k$, therefore $K_{k+2} - e \subseteq G$. Since $L_k(G)$ is connected and k vertices in $K_{k+2} - e$ already attained the maximum possible degree $k + 1$, $L_k(G) = L_k(K_{k+2} - e)$ must hold. But then the number of vertices in $L_k(G)$ is strictly less than the number of vertices in $L(G)$, which is a contradiction.

Subcase C: G contains a K_{k+1} which does not share vertices with any other K_{k+1}

Either this reduces to subcase A or, by connectivity, $k - 1$ vertices of this K_{k+1} belong to a k -clique outside this K_{k+1} . In that case, these $k - 1$ vertices cannot have any further neighbors (since they attained the maximum degree $k + 1$) and there are only two more vertices in the K_{k+1} under consideration. If $k > 3$ then, since $L_k(G)$ is connected, there are no further k -cliques in G and hence $L_k(G)$ is K_{k+1} together with a vertex having exactly two neighbors in K_{k+1} , which is obviously not isomorphic to $L(G)$. If $k = 3$, using the same arguments, we can see that $L_k(G)$ is a K_4 with one or two further vertices having exactly two neighbors in K_4 . In either case, $L_k(G) \neq L(G)$.

Case 3. $\omega(G) = k + 2$

In this case $\Delta(G) \leq k + 1$ together with connectivity implies $G = K_{k+2}$, and this is exactly what had to be proven. \square

7 Concluding remarks

Concerning the algorithmic complexity of recognizing k -line graphs, the problem is trivial for $k = 1$, solvable in linear-time for $k = 2$, and as we have proved, it becomes NP-complete for each $k \geq 3$. It is worth noting that there is a further jump between the behavior of 2- and 3-line graphs, which likely is in connection with the jump occurring in time complexity. Namely, this further difference is in the uniqueness of preimages.

As a matter of fact, each connected line graph different from K_3 has a unique preimage if we disregard isolated vertices. In other words, viewing the situation from the side of preimages, the line graphs of two non-isomorphic graphs containing no isolated vertices and no K_3 -components surely are non-isomorphic. The similar statement is not true for triangle graphs, even if we suppose that every edge of the preimage is contained in a triangle. For example, there are seven essentially different graphs whose triangle graph is the 8-cycle (see Figure 3). Additionally, the number of non-isomorphic pre-images of an n -cycle goes to infinity as $n \rightarrow \infty$ [3].

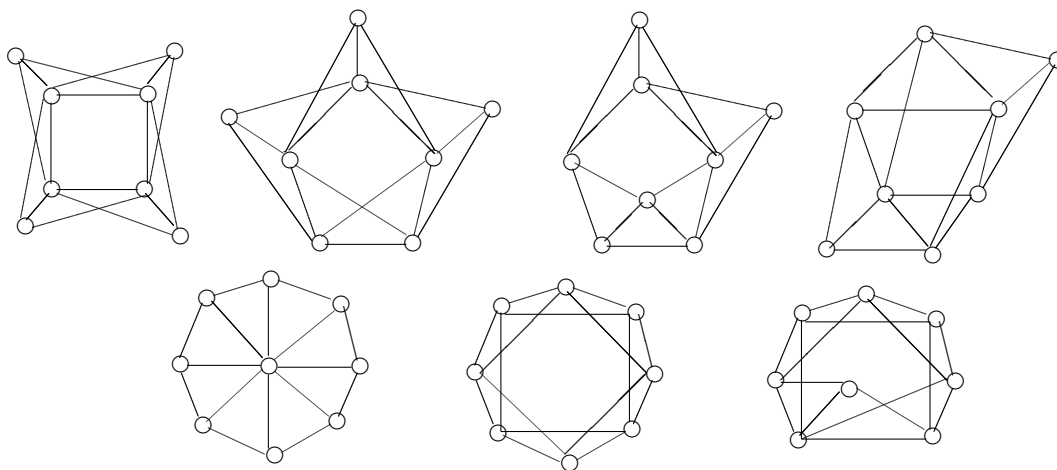


Figure 3: The seven triangle restricted graphs whose triangle graph is C_8 .

Finally, we make some remarks on an open problem closely related to the results of this paper.

For an integer $k \geq 1$, the vertices of the k -Gallai graph $\Gamma_k(G)$ represent the k -cliques of G , moreover its two vertices are adjacent if and only if the corresponding k -cliques of G share a $(k-1)$ -clique but they are not contained in a common $(k+1)$ -clique. By this definition, the 1-Gallai graph $\Gamma_1(G)$ is exactly the complement \overline{G} of G , whilst 2-Gallai graph means Gallai graph in the usual sense, as introduced by Gallai in [11]. Obviously, $\Gamma_k(G) \subseteq L_k(G)$ holds. Moreover $\Delta_k(G)$ and $\Gamma_k(G)$ together determine an edge partition of the k -line graph. Our theorems together with the earlier results from [6] and [2] determine the time complexity of recognition problems of the k -line graphs and the k -anti-Gallai graphs for each fixed $k \geq 1$. Since $G = \Gamma_1(\overline{G})$, every graph is a 1-Gallai graph. But for each $k \geq 2$ this recognition problem remains open.

Problem 21. Determine the time complexity of the recognition problem of k -Gallai graphs for $k \geq 2$.

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